On the topology of Higher-Dimensional Automata

Paris ACTS

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joint work with Thomas Kahl and Rodrigo Lopes
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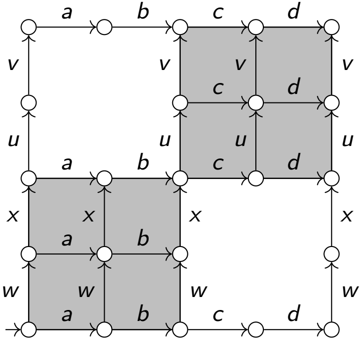
HDAs [van Glabbeek, 2006]

A *higher-dimensional automaton* (HDA) is a tuple $\mathcal{A} = (P_\mathcal{A}, I_\mathcal{A}, \Sigma_\mathcal{A}, \lambda_\mathcal{A})$ where

- $P_\mathcal{A}$ is a precubical set, i.e., a graded set $P_\mathcal{A} = ((P_\mathcal{A})_n)_{n\geq 0}$ with *face maps* $d_i^k\colon (P_\mathcal{A})_n \to (P_\mathcal{A})_{n-1}$ ($n > 0$, $k \in \{0,1\}$, $i \in \{1,\dots,n\}$) satisfying the relations
$$d_i^k \circ d_j^l = d_{j-1}^l \circ d_i^k, \quad i < j;$$
- $I_\mathcal{A} \in (P_\mathcal{A})_0$ is a vertex, called the *initial state*;
- $\Sigma_\mathcal{A}$ is a finite set of *labels*;
- $\lambda_\mathcal{A}\colon (P_\mathcal{A})_1 \to \Sigma_\mathcal{A}$ is a map, called the *labeling function*, such that
$$\lambda_\mathcal{A}(d_i^0 x) = \lambda_\mathcal{A}(d_i^1 x)$$
for all $x \in (P_\mathcal{A})_2$ and $i \in \{1,2\}$.

Higher-dimensional automata



Directed topology: directed spaces

Preordered spaces

A *preordered space* is a topological space X with a preorder relation $\preceq_X$ on it.

D-spaces [Grandis, 2003]

A *d-space* is a topological space X together with a set dX of *directed paths* that contains all constant paths and is closed under concatenation and (not necessarily strictly) increasing partial reparametrization, i.e., precomposition with increasing continuos maps $[0, 1] \to [0, 1]$.

Proposition.

The geometric realization of a precubical set P is naturally a d-space and therefore also naturally a preordered space.

Shared-variable systems

A simple PV-program

- Integer variables: x, y, initially 1 (*mutexes*);
- Two actions: Px decreases x if $x > 0$ and Vx increases x;
- Two copies of the process: $Px; Vx; Py; Vy$.

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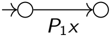

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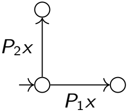

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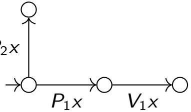

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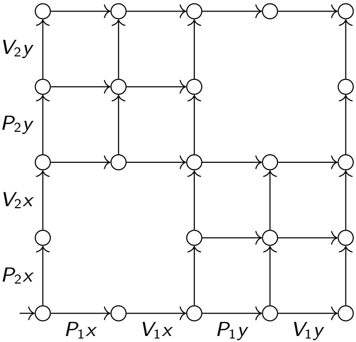

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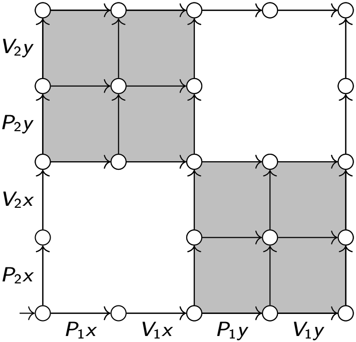

Topology of concurrent systems

[F., Kahl, Lopes, J. Appl. and Comput. Topology, 2025]

Topology of concurrent systems

Question: How complex can the topology of a concurrent system be?

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For every (nonempty) connected polyhedron, there exists a PV-program whose execution space admits a connected component with the same homotopy type as the given polyhedron.

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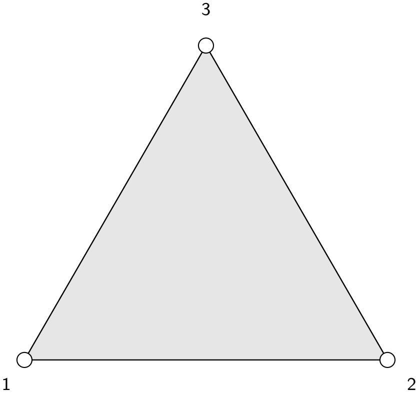
For every (nonempty) connected polyhedron, there exists a shared-variable system whose HDA model has the same homotopy type as the polyhedron.

Topology of concurrent systems



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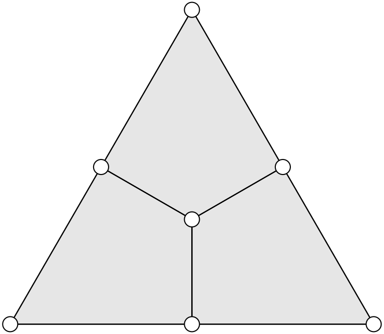

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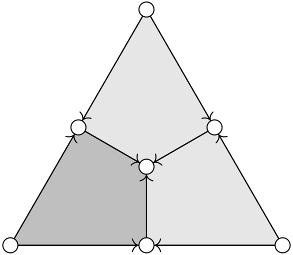

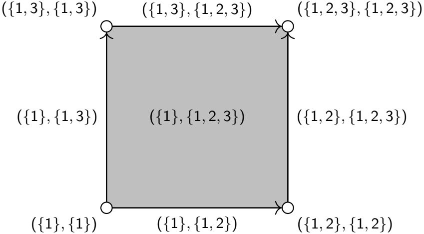

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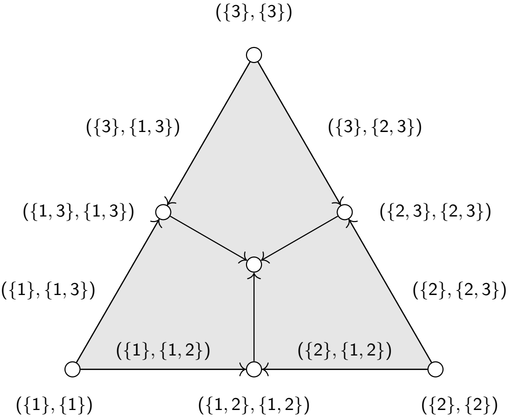

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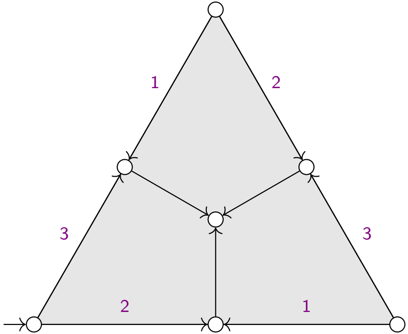

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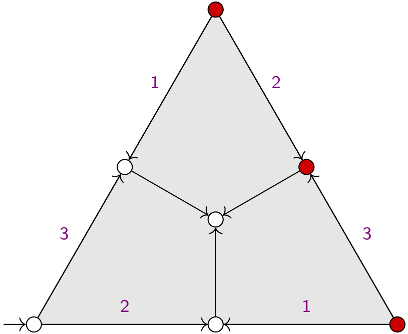

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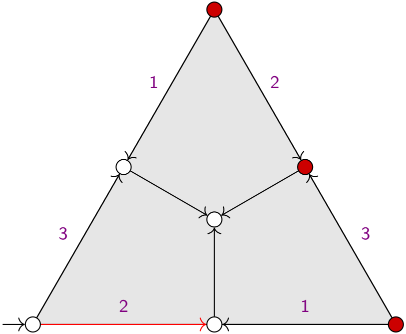

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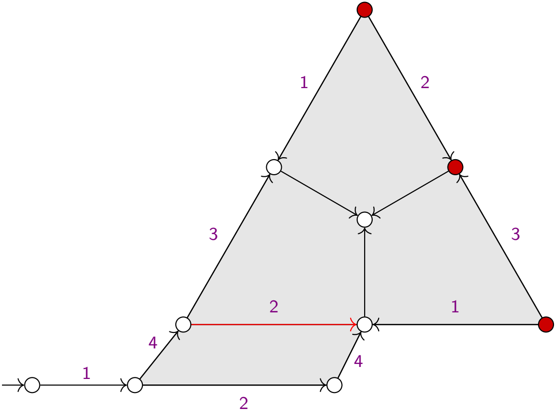

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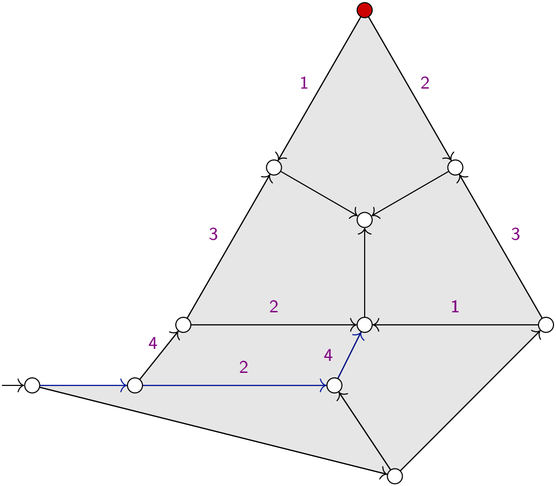

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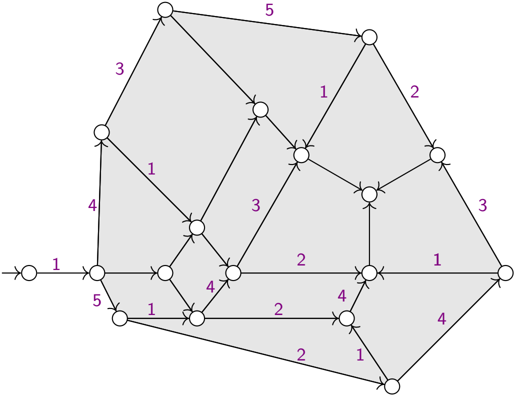


Homology language

Cubical chains and homology

Let P be a precubical set. The *cubical chain complex of* P, $C_*(P)$ is defined in dimension $n \geq 0$ as the free R-module generated by P_n and in dimension $n < 0$ by $C_n(P) = 0$. For $n > 0$, the boundary map $d\colon C_n(P) \to C_{n-1}(P)$ is defined, for a basis element $x \in P_n$, by

$$dx = \sum_{i}^{n} (-1)^i (d_i^0 x - d_i^1 x).$$

For $n \leq 0$, $d = 0$.

The *cubical homology* of P is the ordinary homology of $C_*(P)$, i.e., for $n \in \mathbb{Z}$,

$$H_n(P) = H_n(C_*(P)).$$

Homology language

Homology language [Kahl, 2021]

Let $\mathcal{A}$ be a HDA. The *labeling chain map* $l_{\mathcal{A}}\colon C_*(P_{\mathcal{A}}) \to \Lambda(\Sigma_{\mathcal{A}})$ is defined on basis elements $x \in (P_{\mathcal{A}})_n$ by

$$l_{\mathcal{A}}(x) = \begin{cases} 1_{\Lambda(\Sigma_{\mathcal{A}})}, & n = 0, \\ \lambda_{\mathcal{A}}(e_1^0 x) \wedge \cdots \wedge \lambda_{\mathcal{A}}(e_n^0 x), & n > 0. \end{cases}$$

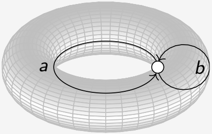
The *homology language* of $\mathcal{A}$ is then the graded submodule

$$\mathcal{HL}(\mathcal{A}) = \operatorname{im} \ell_{\mathcal{A}}$$

of the exterior algebra $\Lambda(\Sigma_{\mathcal{A}})$, where $\ell_{\mathcal{A}}\colon H_*(P_{\mathcal{A}}) \to \Lambda(\Sigma_{\mathcal{A}})$ is the labeling homomorphism induced by the labeling chain map.

Example

Consider the directed torus as an HDA $\mathcal{A}$ with $\Sigma_{\mathcal{A}} = \{a, b\}$.



Then $\mathcal{HL}(\mathcal{A}) = \Lambda(a, b)$, i.e.,

- $\mathcal{HL}(\mathcal{A})_0 = R$,
- $\mathcal{HL}(\mathcal{A})_1 = R \cdot \{a, b\}$,
- $\mathcal{HL}(\mathcal{A})_2 = R \cdot \{a \wedge b\}$,
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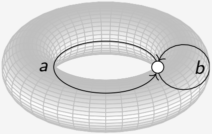

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Properties [Kahl, 2021]

- $\mathcal{HL}(\mathcal{A} \otimes \mathcal{B}) = \mathcal{HL}(\mathcal{A}) \wedge \mathcal{HL}(\mathcal{B})$
- $\mathcal{HL}(\mathcal{A} + \mathcal{B}) = \mathcal{HL}(\mathcal{A}) + \mathcal{HL}(\mathcal{B})$

Homology language

Question: Does the homology language of an HDA inherit some of the algebraic structure of the exterior algebra?

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Diagonal

Let P be a precubical set, $x \in P_n$ and let $I = \{i_1, \dots, i_p\}$ and $I^c = \{j_1, \dots, j_{n-p}\}$ be complementary naturally ordered subsets of $\{1, \dots, n\}$. Define

$$d_I^k x = d_{i_1}^k \dots d_{i_p}^k x$$

and consider the *shuffle permutation* $\sigma_I \in S_n$ which is defined by

$$\sigma_I(r) = \begin{cases} i_r, & 1 \leq r \leq p, \\ j_{r-p}, & p < r \leq n. \end{cases}$$

The *diagonal* of $C_*(P)$ is the natural map $\Delta_{C_*(P)} \colon C_*(P) \to C_*(P) \otimes C_*(P)$ of graded R-modules defined by

$$\Delta_{C_*(P)}(x) = \sum_{I \subseteq \{1,\dots,n\}} \operatorname{sgn}(\sigma_I) d_{I^c}^0 x \otimes d_I^1 x.$$

Homology language

Proposition

The diagonal $\Delta_{C_*(P)}\colon C_*(P) \to C_*(P) \otimes C_*(P)$ is a chain map.

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Proposition

Let $\mathcal{A}$ be an HDA and consider the labeling chain map $l_{\mathcal{A}}\colon C_*(P_{\mathcal{A}}) \to \Lambda(\Sigma_{\mathcal{A}})$. Then $l_{\mathcal{A}}$ is compatible with the diagonal, i.e.,

$$\Delta_{\Lambda(\Sigma)} l_{\mathcal{A}} = (l_{\mathcal{A}} \otimes l_{\mathcal{A}})\Delta_{C_*(P_{\mathcal{A}})}.$$

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Theorem.

The homology language of an HDA is a graded subcoalgebra of the exterior algebra on its alphabet.

Directed homology

Notions of directed homology

- Directed homology [Grandis, 2005]
    - homology equipped with an additive preorder
- Homology graph [Kahl, 2014]
    - homology equipped with a relation
- Natural homology [Dubut, Goubault, Goubault-Larrecq, 2015]
    - natural system of homologies of trace spaces
- . . .

Homology digraph

[F., Kahl, J. Homotopy Relat. Struct., 2024]

Homology digraph

Directional graded vector spaces

A relation $\searrow$ on a graded vector space V is said to be *bilinear* if there exists a graded vector subspace $\mathcal{R} \subseteq V \otimes V$ such that

$$\forall v, w \in V \quad v \searrow w \Leftrightarrow v \otimes w \in \mathcal{R}.$$

A *directional graded vector space* is a graded vector space V equipped with a bilinear relation $\searrow_V$.

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The homology digraph of a pair of preordered spaces

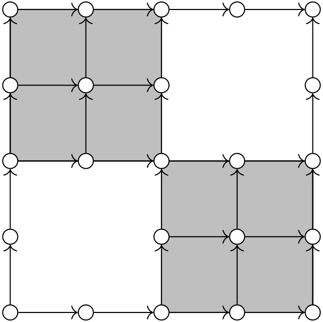
The *homology digraph* of the pair (X, A) is the directional graded vector space $H_*(X, A)$ where the bilinear relation is generated by the relation $\nearrow$ defined as follows: given relative homology classes $\alpha, \beta \in H_*(X, A)$ (of possibly different degrees), we set $\alpha \nearrow \beta$ if there exist subspaces $E, F \subseteq X$ such that

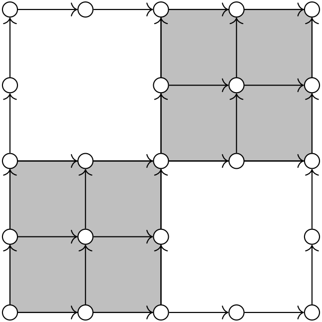
- $\alpha \in \operatorname{im} H_*((E, E \cap A) \hookrightarrow (X, A))$;
- $\beta \in \operatorname{im} H_*((F, F \cap A) \hookrightarrow (X, A))$;
- $\forall x \in E, y \in F \quad x \preceq_X y$.

The *homology digraph* of X is defined to be the homology digraph of the pair $(X, \emptyset)$.

Homology digraph

Homeomorphic HDA models of simple PV-programs with different homology digraphs





Homology digraph

Dihomotopy invariance

If $f\colon X \to Y$ is a monotone continuous map that is a homotopy equivalence with monotone homotopy inverse, then $f_*\colon H_*(X) \to H_*(Y)$ is an isomorphism of directional graded vector spaces.

Exact sequences

Let (X, A) be a pair of preordered spaces. The connecting homomorphisms $H_k(X, A) \to H_{k-1}(A)$ of the long exact sequence of the pair (X, A) constitute a morphism of directional graded vector spaces.

Excision

Let (X, A) be a pair of preordered spaces, and let U be a subset of A such that $\overline{U}^X \subseteq \mathrm{int}_X(A)$. Then the inclusion induces an isomorphism of directional graded vector spaces $H_*(X \setminus U, A \setminus U) \to H_*(X, A)$.

Homology digraph

Coproduct

Let $((X_i, A_i))_{i\in\mathcal{I}}$ be a family of pairs of preordered spaces. Then the inclusions $\iota_j\colon (X_j, A_j) \hookrightarrow \coprod_{i\in\mathcal{I}}(X_i, A_i)$ induce an isomorphism of directional graded vector spaces

$$\bigoplus_{i\in\mathcal{I}} H_*(X_i, A_i) \to H_*(\coprod_{i\in\mathcal{I}} X_i, \coprod_{i\in\mathcal{I}} A_i).$$

Wedge

Let X and Y be preordered spaces, and let $x_0 \in X$ and $y_0 \in Y$ be minimal elements such that the inclusions $\{x_0\} \hookrightarrow X$ and $\{y_0\} \hookrightarrow Y$ are closed cofibrations. Then the inclusions $(X, x_0) \hookrightarrow (X \vee Y, x_0 = y_0)$ and $(Y, y_0) \hookrightarrow (X \vee Y, x_0 = y_0)$ induce an isomorphism of directional graded vector spaces

$$H_*(X, x_0) \oplus H_*(Y, y_0) \to H_*(X \vee Y, x_0 = y_0).$$

Homology digraph

Künneth theorem

For any two preordered spaces X and Y, the cross product homomorphism

$$\times\colon H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$$

is an isomorphism of directional graded vector spaces.

Thank you!